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An Overview of the Common Fluid Models Used in Fluid-Structure Interactions

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This report derives and discusses some of the fluid models more commonly used in fluid-structure interaction problems. The emphasis is on linearized, self-contained governing equations on the solid-fluid boundary which directly govern the fluid pressure response there. This report is intended to aid engineers specialized in the mechanics of solids who find themselves in the position of having to model the response of a solid structure which is interacting with an adjacent fluid. The overview given should enable such an engineer to obtain at least a cursory understanding of the origin and limitations of the fluid models which are in common use for such purposes. The report starts from a general model of fluid mechanics and moves to an acoustical medium model from which the retarded potential formulation is obtained. This then leads to the early-time, late-time, and Doubly Asymptotic Approximation models.

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AN OVERVIEW OF THE COMMON FLUID MODELS USED IN FLUID-STRUCTURE INTERACTIONS

INTRODUCTION

There are occasions when engineers specialized in the mechanics of solids find themselves in the position of having to model the response of a solid structure which is interacting with an adjacent fluid whose density is not negligible in comparison to that of the solid. The loading on the solid by the fluid must then be properly taken into account in order to accurately predict the response of the solid, but there is a lack of background on the part of the engineer as to what is an appropriate fluid model to couple to the one that he is already using for the solid. The overview given by this report should enable such an engineer to obtain at least a cursory understanding of the origin and limitations of the fluid models which are in common use for such purposes. With this as a base, the references given should allow one to pursue both greater details and actual computer code implementations.

This report derives and discusses some of the fluid models more commonly used in fluid-structure interaction problems. The emphasis here is on linearized equations governing disturbances generated in, and propagated through, the fluid since this forms such a large part of the fluid-structure interaction literature. The fluid may either surround the solid, be contained by the solid in an internal cavity, or both. In many of the cases in which the fluid surrounds the solid the fluid is idealized as extending to infinity. This is a source of difficulty in numerically modeling the physical system in such cases since one cannot directly extend the computational grid to infinity. Accurate truncation to a finite volume of fluid would create external boundary condition problems. A common approach to circumventing this problem in the linearized case is to reformulate the fluid model into self-contained equations on the solid-fluid boundary which directly govern the fluid pressure response there. Several versions of this approach are discussed in this report. They are all based on a parent fluid model which is derived and discussed next.

FLUID MECHANICS FORMULATION

The behavior of the fluid in fluid-structure interaction problems is governed by the mass conservation equation

$$D\rho + \rho\frac{\partial u_j}{\partial x_j} = 0\,, \tag{1}$$

the momentum conservation equation

$$\rho D u_i = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j}\,, \tag{2}$$

and the energy conservation equation

$$\rho D\left(e + \frac{u^2}{2}\right) = \rho u_i F_i + \frac{\partial}{\partial x_j}(u_i \sigma_{ij}) - \frac{\partial q_j}{\partial x_j}\,, \tag{3}$$

where the usual Einstein convention of summation over repeated subscripts is assumed, and where

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ρ = fluid mass density,

u_i = fluid velocity vector,

F_i = body force per unit mass,

σ_{ij} = stress tensor,

e = internal energy per unit mass

q_i = heat flux vector

$u^2 = u_i u_i$

$D = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$,

t = time, and

x_i = spatial position vector.

The "material derivative" D can be interpreted as a time derivative following the motion of the fluid because

$$\frac{d}{dt}\int_V \rho\theta dV = \int_V \rho D\theta dV$$

holds for θ any extensive property of the fluid and V the volume of any fluid particle of fixed mass moving with the flow. Taking the vector inner product of u_i with (2) and subtracting the results from (3) gives the alternative energy equation

$$\rho De = \sigma_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}. \tag{4}$$

The above governing equations must be supplemented by appropriate equations-of-state, by appropriate constitutive relations, and by appropriate boundary conditions. The most important one at the solid-fluid interface consists of the no-slip, no-penetration boundary condition

$$u_i = (u_i)_{solid} \tag{5}$$

at each point of the interface at each instant of time. The stress tensor takes the form

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \tag{6}$$

for isotropic fluids, where

p = (thermodynamic) fluid pressure and

d_{ij} = "viscous" part of the stress tensor.

For Newtonian fluids the "viscous" part of the stress tensor is linearly related to the spatial derivatives of the fluid velocity as

$$d_{ij} = \eta \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \left(\frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \right),$$

where

μ = (shear) viscosity and

η = expansion (or bulk) viscosity.

Substituting (6) into (4) and then using (1) leads to the

$$\rho De - \left(\frac{p}{\rho}\right) D\rho = d_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} \tag{7}$$

version of the energy equation.

The nonequilibrium setting generally present in a flowing fluid requires appropriate interpretations [1, pp. 151-152] for thermodynamic variables such as entropy. With this understanding, the thermodynamic relations

$$Dp = c^2 D\rho + \left(\frac{\partial p}{\partial s}\right)_\rho Ds \tag{8}$$

and

$$TDs = De + pD\left(\frac{1}{\rho}\right) \tag{9}$$

are available, for which

$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ = local speed of sound squared,

s = entropy per unit mass, and

T = absolute temperature.

Equation (9) is the well known Gibbs relation for a simple compressible substance. The thermodynamic relation [1, p. 170]

$$\left(\frac{\partial p}{\partial s}\right)_\rho = \frac{\rho c^2 \beta T}{c_p},$$

where

β = coefficient of thermal expansion and

c_p = specific heat (at constant pressure),

combines with (8) to give

$$D\rho = c^{-2}Dp - \frac{\rho\beta T}{c_p}Ds\,. \tag{10}$$

Combining (1) and (10) leads to

$$c^{-2}Dp + \rho\frac{\partial u_j}{\partial x_j} = \frac{\rho\beta T}{c_p}Ds\,. \tag{11}$$

Combining (2) and (6) leads to

$$\rho Du_i + \frac{\partial p}{\partial x_i} = \rho F_i + \frac{\partial d_{ij}}{\partial x_j}\,. \tag{12}$$

Combining (7) and (9) leads to

$$\rho TDs = d_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}\,. \tag{13}$$

Equations (11), (12), and (13), which govern the fluid behavior, are still quite general.

The first assumption usually made for fluid-structure interaction problems is that dissipative and diffusive-type processes (of molecular transport origin), which lead to internal irreversibilities, are negligible. This translates into the specific conditions

$$q_i = 0$$

and

$$d_{ij} = 0\,,$$

which represent an inviscid-fluid, negligible-heat-conduction postulate. Implementing these conditions in (13) leads to

$$Ds = 0\,. \tag{14}$$

Using (14) in (11) and the inviscid-fluid condition $d_{ij} = 0$ in (12) gives

$$Dp + \rho c^2\frac{\partial u_j}{\partial x_j} = 0 \tag{15}$$

and

$$\rho Du_i + \frac{\partial p}{\partial x_i} = \rho F_i \tag{16}$$

respectively. Equations (15) and (16) represent the governing equations for the fluid behavior under the given assumptions. One problem which originates from the inviscid assumption is that (5) can no longer be satisfied in its entirety in conjunction with (15) and (16). This issue will be addressed more fully later.

The fluid behavior in fluid-structure interaction problems is most often thought of in terms of disturbances in the fluid which propagate through it. The fluid models can be categorized according to whether the disturbances are of small amplitude, as is the case to be considered in this report, or of finite amplitude [2, 3]. An additional consideration is whether the disturbances are produced by, or interact significantly

with, a background flow field through which they propagate [4]. The widely-used fluid model emphasized in this report assumes that the total flow field decomposes into a steady, mean (background) flow field of slowly varying velocity and small-amplitude disturbances superimposed on this mean flow. This decomposition takes the form

$$p = p_0 + p_1 \tag{17}$$

$$\rho = \rho_0 + \rho_1 \tag{18}$$

$$u_i = u_{i0} + u_{i1} \tag{19}$$

$$c^2 = c_0^2 + c_1^2 , \tag{20}$$

where those variables labeled with a 0 subscript are associated with the mean flow and those labeled with a 1 subscript are associated with the small-amplitude disturbances and hence they have small magnitudes. The steady mean flow assumption can be expressed as

$$\frac{\partial F_i}{\partial t} = 0 \tag{21}$$

$$\frac{\partial u_{i0}}{\partial t} = 0 \tag{22}$$

$$\frac{\partial p_0}{\partial t} = 0 , \tag{23}$$

whereas the slowly varying velocity assumption for the mean flow takes the form

$$\frac{\partial u_{i0}}{\partial x_j} = 0 , \tag{24}$$

at least to first order. The mean flow is independent of the disturbances and, in particular, it exists even in their absence. This means that the mean flow field must separately satisfy (15) and (16). Substituting (21) through (24) into these equations leads to

$$u_{j0}\frac{\partial p_0}{\partial x_j} = 0$$

and

$$\frac{\partial p_0}{\partial x_i} = \rho_0 F_i . \tag{25}$$

These two equations combine to give the relation

$$\rho_0 u_{i0} F_i = 0 . \tag{26}$$

Inserting (17) through (20) into (15) and (16), neglecting second order terms, and utilizing (22) through (26) leads to the linearized equations

$$D_0 p_1 + \rho_0 c_0^2 \frac{\partial u_{j1}}{\partial x_j} = -\rho_0 u_{j1} F_j \tag{27}$$

and

$$\rho_0 D_0 u_{i1} + \frac{\partial p_1}{\partial x_i} = \rho_1 F_i \,, \tag{28}$$

where

$$D_0 = \frac{\partial}{\partial t} + u_{j0} \frac{\partial}{\partial x_j}$$

is the material derivative for the mean flow.

The next simplifying assumption consists of the

$$\frac{LF}{c_0^2} \ll 1 \tag{29}$$

constraint on the magnitude of the body force, where

F = a representative magnitude of the body force per unit mass and

L = a representative length scale of the structure.

The constraint (29) is certainly true for the most common case for which the body force is gravity [1, p. 166]. The pressure gradient term in (28) is of the order [5, p. 245]

$$\|\nabla p_1\| \approx O\left(\frac{c_0^2 \rho_1}{L}\right)$$

in magnitude and hence the estimate

$$\frac{\|\rho_1 \vec{F}\|}{\|\nabla p_1\|} \approx O\left(\frac{LF}{c_0^2}\right)$$

follows for equation (28). (The $\|\vec{A}\|$ symbol represents the norm of $\vec{A}$ for any given $\vec{A}$ with components A_i. In this case it is the Euclidean norm so that it represents the magnitude of the vector.) The estimate

$$\frac{\rho_0 u_{j1} F_j}{\left(\rho_0 c_0^2 \frac{\partial u_{j1}}{\partial x_j}\right)} \approx O\left(\frac{LF}{c_0^2}\right)$$

is also obtained, for equation (27), so that (27) and (28) reduce to

$$D_0 p_1 + \rho_0 c_0^2 \frac{\partial u_{j1}}{\partial x_j} = 0 \tag{30}$$

and

$$\rho_0 D_0 u_{i1} + \frac{\partial p_1}{\partial x_i} = 0 \tag{31}$$

under the constraint of (29). Equations (30) and (31), which govern the fluid behavior under all of the above assumptions, are valid for an inhomogeneous fluid for which ρ_0 and c_0^2 are functions of position.

Equation (31) can be integrated easily for the common case in which ρ_0 is taken to be a constant and the gradient of a velocity potential ϕ_1 is utilized for u_{i1}. This assumption, along with the assumption of constant c_0^2, will be taken as valid for the remainder of the report. This leads to

$$\rho_0 D_0 \phi_1 + p_1 = f(t)$$

as the integration of (31) for an arbitrary function $f(t)$ of time only. There is no loss in generality in taking f to be zero since any nonzero f can be absorbed into the definition of ϕ_1 as

$$(\phi_1)_{new} = \phi_1 - \rho_0^{-1} \int_0^t f(t')\, dt'$$

without affecting u_{i1}. The above expression hence reduces to

$$p_1 = -\rho_0 D_0 \phi_1 \tag{32}$$

with the velocity potential defined by

$$u_{i1} = \frac{\partial \phi_1}{\partial x_i} . \tag{33}$$

Equation (32) can be substituted into (30) to give

$$D_0^2 \phi_1 - c_0^2 \frac{\partial^2 \phi_1}{\partial x_j \partial x_j} = 0 \tag{34}$$

which, with appropriate boundary conditions, can be solved for the disturbance solution ϕ_1. The complete model for the fluid disturbance is hence given, at this point, by equations (32) through (34) and appropriate boundary conditions. Operating on (34) with $-\rho_0 D_0$ and using (32) leads to

$$D_0^2 p_1 - c_0^2 \frac{\partial^2 p_1}{\partial x_j \partial x_j} = 0 \tag{35}$$

as the equation for p_1.

Classical Wave Equation of Acoustics

When the above equations are transformed into a reference frame which is at rest with respect to the fluid they then describe the fluid as an acoustical medium whose behavior is governed by the classical wave equation. This is the simplest, and probably the most common, rigorous fluid model in use for fluid-structure interaction problems. It, and approximations to it, will form the basis of the remainder of the report. The simplest way to obtain the version of the above fluid model in a frame at rest with respect to the fluid is to take $u_{i0} = 0$ in all of the above expressions so that D_0 reduces to $\partial/\partial t$. To ease the notational burden, the subscript 1 on p_1, ϕ_1, and u_{i1} will be dropped. In addition, since ρ_1 and c_1 are never referred to, the subscript 0 on ρ_0 and c_0 will be dropped so that ρ will represent the constant ρ_0 and c will represent the constant c_0. The p of (17) will be denoted by p_{total} so that (17) becomes

$$p_{total} = p_0 + p \,. \tag{36}$$

Equations (23) and (25) are repeated here as

$$\frac{\partial p_0}{\partial t} = 0 \,, \tag{37}$$

and

$$\frac{\partial p_0}{\partial x_i} = \rho F_i \tag{38}$$

under this new notation. (Equation (21) is not needed since it is implied by (37), (38), and the fact that ρ is a positive constant.) Equations (32), (33), and (34), which govern the fluid disturbance propagation, become

$$p = -\rho \frac{\partial \phi}{\partial t} \,, \tag{39}$$

$$u_i = \frac{\partial \phi}{\partial x_i} \,, \tag{40}$$

and

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x_j \partial x_j} = 0 \,. \tag{41}$$

As in the case for ϕ, the classical wave equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x_j \partial x_j} = 0 \,, \tag{42}$$

results, in this case from (35), for the acoustic pressure p. The disturbance part of the total mass density also obeys the same wave equation, but it is not usually explicitly used in modeling the fluid.

Other than boundary conditions, which will be discussed in the next subsection, equations (36) through (41) give a complete model, henceforth referred to as the potential-based model, of the fluid behavior. Equation (42) is an equivalent substitution for (41) in the above complete set, for $t \geq t_0$, if the initial condition

$$\left(\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x_j \partial x_j} \right)_{t = t_0} = 0 \tag{43}$$

is imposed on ϕ. This is most often satisfied by the condition that ϕ is a constant, usually zero, at some time $t = t_0$ and earlier. An alternative fluid model, henceforth called the pressure-based model, is hence given by equations (36) through (40), (42), and (43). The solution to the governing equations ultimately provides a history of the distribution of p_{total} along the solid-fluid interface, which is required for predicting the response of the solid.

Boundary Conditions

Either of the above alternative, equivalent models is incomplete without appropriate boundary conditions. The boundary condition for p_0 is simply

$$p_0(x_{i0}) = p_{00}\,, \tag{44}$$

where

x_{i0} = a given, fixed point in the fluid or on its boundary and

p_{00} = a prescribed constant value.

Both of the alternative models are based on solving the classical wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x_j \partial x_j} = 0\,, \tag{45}$$

where ψ is either ϕ or p, depending on the model. The appropriate boundary information for this equation is to specify either ψ (Dirichlet), $\partial\psi/\partial n$ (Neumann), or a linear combination (mixed) of them on the boundary, where

$$\frac{\partial}{\partial n} = n_j \frac{\partial}{\partial x_j} \text{ and}$$

n_i = the unit normal vector on the boundary, specified as either inward or (usually) outward.

In addition to this, one needs to specify an initial condition for $\psi(x_i, t)$ as

$$\psi(x_i, 0) = \psi_0(x_i)\,, \tag{46}$$

where

$\psi_0(x_i)$ is the prescribed intial distribution of ψ.

In each model the closest approximation that one can make to (5) at the solid-fluid interface requires the Neumann boundary condition. Taking the ordinary vector inner product of (5) with n_i gives the result

$$u_n = (u_n)_{solid}\,, \tag{47}$$

where

$u_n = n_j u_j$ is defined as the normal component of the velocity vector.

Taking $\partial/\partial t$ of (5) and then taking the vector inner product of the results with n_i leads to

$$a_n = (a_n)_{solid}\,, \tag{48}$$

where

$a_n = n_j \frac{\partial u_j}{\partial t}$ is defined as the normal component of the acceleration vector.

Both (47) and (48) are only approximations to (5) since they each contain only part of the information content of (5). Equations (47) and (48) are not equivalent as can be seen from the identity

$$\frac{\partial u_n}{\partial t} = u_j \frac{\partial n_j}{\partial t} + a_n . \tag{49}$$

The vector $\partial n_i/\partial t$ is easily seen to be tangential because

$$n_j \frac{\partial n_j}{\partial t} = \frac{\partial}{\partial t}\left(\frac{n_j n_j}{2}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\right) = 0 .$$

If $\partial n_i/\partial t$ is negligible or zero then taking $\partial/\partial t$ of (47) gives (48), and integrating (48) over time from t_0 to t gives (47) for $t > t_0$ if (47) is true at $t = t_0$. This is at least approximately the case when the displacements of the solid during its response are small. The relation (47) is used for the potential-based fluid model as can be seen by taking the vector inner product of (40) with n_i and substituting (47) into the results to get

$$\frac{\partial \phi}{\partial n} = (u_n)_{solid} \tag{50}$$

as the appropriate fluid-solid interface boundary condition. Taking the gradient of (39), taking the vector inner product of the results with n_i, and substituting (48) into the results leads to

$$\frac{\partial p}{\partial n} = -\rho (a_n)_{solid} \tag{51}$$

as the appropriate fluid-solid interface boundary condition for the pressure-based fluid model. The two models hence lead to different results when the unit normal at the interface is not constant in time.

In many of the problems in fluid-structure interactions the fluid surrounds the solid and goes off to infinity in all directions. In these cases one needs to consider an additional boundary condition which applies as one approaches infinity. A radiation boundary condition, which will be discussed later where its meaning will be easier to understand, is usually appropriate.

REFORMULATION INTO INTERFACE INTEGRAL EQUATION

The source of the disturbances propagating through the fluid may be the solid itself, or the source may be located within the fluid, or both. The modeling of incident pressure waves which are emanating from a source within the fluid and impinging on the solid can be handled by including a source term σ (of bounded extent) in the governing wave equation (45) for the disturbances, so that

$$\frac{\partial^2 \psi}{\partial x_j \partial x_j} - c^{-2} \frac{\partial^2 \psi}{\partial t^2} = \sigma \tag{52}$$

governs the disturbances for ψ either the velocity potential or the acoustic pressure, where $\sigma = 0$ throughout most of the fluid. Let $\bar{\psi}$ denote the Fourier transform, with respect to time, of ψ, so that

$$\bar{\psi}(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(t) \exp(i\omega t)\, dt$$

and similarly for any other dependent variables. As in [6], equation (52) will be reformulated into a surface integral equation through the intermediate use of Fourier transforms. Taking the transform of (52) and using

$$\overline{\frac{\partial \psi}{\partial t}} = -i\omega\bar{\psi}(\omega) \tag{53}$$

leads to

$$\frac{\partial^2 \bar{\psi}}{\partial x_j \partial x_j} + k^2 \bar{\psi} = \bar{\sigma}\,, \tag{54}$$

where

$k = \omega/c$ defines k.

An equation governing the solution to (54) on the surface bounding the fluid can be obtained through the use of a Green's function G defined by

$$\left(\frac{\partial^2}{\partial x_j \partial x_j} + k^2\right) G(x_i, x'_i) = \delta(x_i - x'_i)\,, \tag{55}$$

where

δ = the Dirac delta function, (actually a generalized function or distribution).

For the case where (55) is solved for an infinite space one gets the solution

$$G(x_i, x'_i) = -\frac{exp(ikr)}{4\pi r}\,, \tag{56}$$

where r_i and r are defined by

$$r_i = x_i - x'_i \tag{57}$$

$$r = (r_j r_j)^{1/2}\,. \tag{58}$$

The two identities

$$\bar{\psi}\frac{\partial^2 G}{\partial x_j \partial x_j} - G\frac{\partial^2 \bar{\psi}}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(\bar{\psi}\frac{\partial G}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(G\frac{\partial \bar{\psi}}{\partial x_j}\right)$$

and

$$\bar{\psi}\frac{\partial^2 G}{\partial x_j \partial x_j} - G\frac{\partial^2 \bar{\psi}}{\partial x_j \partial x_j} = \bar{\psi}\delta - G\bar{\sigma}\,,$$

which follows from (54) and (55), can be combined to give

$$\bar{\psi}\delta = G\bar{\sigma} + \frac{\partial}{\partial x_j}\left(\bar{\psi}\frac{\partial G}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(G\frac{\partial \bar{\psi}}{\partial x_j}\right)\,. \tag{59}$$

Let S_R denote the surface of a sphere of radius R whose fixed center lies within the solid. The radius R is taken to be large (ultimately infinity) so that the surface S_R is completely within the fluid and it completely

contains all of the source $\bar{\sigma}$. (The source $\bar{\sigma}$ is zero outside of S_R.) Let S denote the solid-fluid interface surface which encloses the solid. Let S_T denote the surface defined by

$$S_T = S \cup S_R$$

and denote the fluid volume which S_T encloses by V. Integrating (59) over V leads to

$$\int_V \bar{\psi}\delta dV' = \int_V G\bar{\sigma}dV' + \int_{S_T} (\bar{\psi}\frac{\partial G}{\partial n'} - G\frac{\partial \bar{\psi}}{\partial n'})\, dS' , \tag{60}$$

where the unit normal in the surface integral is outward with repect to the fluid, upon using Gauss' theorem, which states that

$$\int_V \frac{\partial}{\partial x'_j} A_j(x'_i)\, dV' = \int_S n'_j A_j(x'_i)\, dS'$$

for reasonably-behaved, but otherwise arbitrary A_i and for any (reasonable) given volume V which is enclosed by surface S with outward unit normal n_i. Let V_{source} denote the volume of the source $\bar{\sigma}$, so that $\bar{\sigma}$ is only nonzero within V_{source}. Splitting S_T into its component surfaces S_R and S, equation (60) becomes

$$\int_V \bar{\psi}\delta dV' = \int_{V_{source}} G\bar{\sigma}dV' + \int_S (\bar{\psi}\frac{\partial G}{\partial n'} - G\frac{\partial \bar{\psi}}{\partial n'})\, dS' + \int_{S_R} (\bar{\psi}\frac{\partial G}{\partial n'} - G\frac{\partial \bar{\psi}}{\partial n'})\, dS' . \tag{61}$$

The radiation boundary condition, with G from (56), leads to the vanishing of the S_R integral in (61) as R approaches infinity. One formulation of the radiation boundary condition is to take the $\bar{\psi}$ on S_R to be a purely outgoing wave of the form

$$\bar{\psi} = \psi_0(\theta, \phi)\frac{exp(ikR)}{R} , \tag{62}$$

where ψ_0 satisfies

$$\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial \psi_0}{\partial \theta}) + \frac{1}{\sin\theta}\frac{\partial^2 \psi_0}{\partial \phi^2} = 0$$

so that $\bar{\psi}$ satisfies the homogeneous ($\bar{\sigma}$ equal zero) version of (54),

$R_i = x'_i - x_{i0}$,

$R = (R_j R_j)^{1/2}$,

x'_i is on S_R

x_{i0} is the center of the sphere S_R , and

θ and ϕ are the spherical polar angles specifying the direction of R_i .

(The equation for ψ_0 can be transformed into Laplace's equation in μ and ϕ by the substitution $\mu = \log[(1-\cos\theta)/\sin\theta]$.) For large R one finds that

$$r \approx R\left[1 - \frac{\varepsilon_j R_j}{R^2} + \ldots\right]$$

and

$$r^{-1} \approx R^{-1}\left[1 + \frac{\varepsilon_j R_j}{R^2} + \ldots\right]$$

to first order in ε/R, where

$$\varepsilon_i = x_i - x_{i0} ,$$

$$\varepsilon = (\varepsilon_j \varepsilon_j)^{1/2} ,$$

$$r_i = \varepsilon_i - R_i \text{ for the } r_i \text{ of (57),}$$

$$x_i \text{ is on } S ,$$

$$r = (r_j r_j)^{1/2} = R\left[1 + \left(\frac{\varepsilon^2 - 2\varepsilon_j R_j}{R^2}\right)\right]^{1/2} , \text{ and}$$

$$\frac{\varepsilon}{R} \ll 1 .$$

These approximations to r and r^{-1} are substituted into (56) and the resulting G, along with (62), are then substituted into the S_R integral term of (61). It is easily seen that the lowest order ($r \approx R$) approximation gives identically zero for this S_R integral term and hence the first order terms must be retained. The result is that the S_R integral in (61) is of order $1/R$ so that the integral vanishes as R goes to infinity.

The $\bar{\sigma}$ term in (61) represents the waves emanating from the (transformed) source $\bar{\sigma}$ in an infinite-space fluid, that is, in the absence of the solid. As such, it represents the incident part of the wave which impinges on the solid from sources internal to the fluid. It is hence relabeled as

$$\bar{\psi}_{inc} = \int_{V_{source}} G\bar{\sigma} dV' \tag{63}$$

where the subscript inc denotes the incident part of the wave. Taking the inverse transform of (63) and using the relation

$$\overline{f(t-a)} = exp(i\omega a)\overline{f(\omega)} \tag{64}$$

for arbitrary (reasonably behaved) f, with $f = \sigma$ and $a = r/c$ since $k = \omega/c$, leads to the well known infinite space result

$$\psi_{inc}(x_i, t) = -\frac{1}{4\pi}\int_{V_{source}} \frac{\sigma(x'_i, t - r/c)}{r} dV' \tag{65}$$

upon using (56), where r is given by (58). The ψ_{inc} of (65) satisfies (52) in an infinite fluid (no solid).

When the radiation boundary condition and (63) are inserted into (61) and R is then taken to infinity one gets the result

$$\int_V \bar{\psi}\delta dV' = \bar{\psi}_{inc} + \int_S (\bar{\psi}\frac{\partial G}{\partial n'} - G\frac{\partial \bar{\psi}}{\partial n'}) dS' .$$

Substituting (56) into this result gives

$$\int_V \bar{\psi}\delta dV' = \bar{\psi}_{inc} + \frac{1}{4\pi}\int_S \{r^{-3}n'_j r_j (ikr-1)\,\bar{\psi} + r^{-1}\frac{\partial\bar{\psi}}{\partial n'}\}\,exp\,(ikr)\,dS'$$

upon using

$$\frac{\partial r}{\partial n'} = -\frac{n'_j r_j}{r}\,, \tag{66}$$

where r_i and r are defined by (57) and (58) respectively. Switching to an inward unit normal vector with respect to the fluid gives

$$\int_V \bar{\psi}\delta dV' = \bar{\psi}_{inc} + \frac{1}{4\pi}\int_S \{r^{-3}n'_j r_j (1-ikr)\,\bar{\psi} - r^{-1}\frac{\partial\bar{\psi}}{\partial n'}\}\,exp\,(ikr)\,dS'\,, \tag{67}$$

where the n'_i vector is now outward with respect to the solid. The delta function, at least the specific one defined by (55) through (58), has the property that

$$\int_V f(x'_i)\,\delta\,(x_i - x'_i)\,dV' = \begin{cases} f(x_i) \text{ for } x_i \text{ inside } V \\ 0 \text{ for } x_i \text{ outside } V \\ \dfrac{\Omega}{4\pi} f(x_i) \text{ for } x_i \text{ on } S \end{cases} \tag{68}$$

for continuous functions f, where S is the surface enclosing the given volume V and

Ω = (the solid angle subtended by the surface S at x_i) = 2π for smooth surfaces.

(The first two properties are universal for all delta functions.) Using $\bar{\psi}$ for f in (68) and substituting the results into (67) gives

$$\bar{\psi}\,(x_i, k) = 2\bar{\psi}_{inc} + \frac{1}{2\pi}\int_S \{r^{-3}n'_j r_j (1-ikr)\,\bar{\psi} - r^{-1}\frac{\partial\bar{\psi}}{\partial n'}\}\,exp\,(ikr)\,dS' \tag{69}$$

as the governing integral equation for $\bar{\psi}$ on S assuming that S is smooth,

$$\bar{\psi}\,(x_i, t) = \bar{\psi}_{inc} + \frac{1}{4\pi}\int_S \{r^{-3}n'_j r_j (1-ikr)\,\bar{\psi} - r^{-1}\frac{\partial\bar{\psi}}{\partial n'}\}\,exp\,(ikr)\,dS' \tag{70}$$

as the equation for finding $\bar{\psi}$ within the fluid after having first found it on S from solving (69), and

$$0 = \bar{\psi}_{inc} + \frac{1}{4\pi}\int_S \{r^{-3}n'_j r_j (1-ikr)\,\bar{\psi} - r^{-1}\frac{\partial\bar{\psi}}{\partial n'}\}\,exp\,(ikr)\,dS' \tag{71}$$

as the equation for x_i within the solid. Equation (71) serves as an alternative to (69) for determining $\bar{\psi}$ on S. The transform of (50) with $\bar{\psi} = \bar{\phi}$ for the potential-based model, or the transform of (51) with $\bar{\psi} = \bar{p}$ for the pressure-based model, are substituted for $\partial\bar{\psi}/\partial n'$ in (69) to obtain a governing integral equation for the fluid response on S which includes the boudary conditions.

Solving the Equations in Frequency Space

In some of the cases for which either the solid constitutive behavior is linear or the solid behavior is prescribed it is useful to solve for the fluid response in the frequency space setting represented by (69) and the transform of either (50) or (51). It is well known [7], however, that (69) with prescribed $\partial\overline{\psi}/\partial n'$ and zero $\overline{\psi}_{inc}$ does not yield a unique solution at certain discrete values of k (that is, characteristic frequencies, since $k = \omega/c$). Reference [7] offers the remedy of using a discretized version of (71) at selected locations to supplement the main model, a discretized version of (69), so as to form an overdetermined system which is solved by least squares. This approach,which is implemented in the computer code CHIEF, is able to provide accurate solutions even at the problem frequencies. Alternative approaches exist [8 and 9] for overcoming the same problem, such as that embodied in the computer code CONDOR.

Time Domain (Retarded) Formulation

Equation (69) can be reformulated into

$$\overline{\psi}(x_i, k) = 2\overline{\psi}_{inc} + \frac{1}{2\pi}\int_S \{r^{-3} n'_j r_j (\overline{\psi} + \frac{r}{c}\frac{\overline{\partial\psi}}{\partial t}) - r^{-1}\frac{\partial\overline{\psi}}{\partial n'}\} exp(ikr)\, dS'$$

using (53). The inverse transform of this equation gives

$$\psi(x_i, t) = 2\psi_{inc} + \frac{1}{2\pi}\int_S \{r^{-3} n'_j r_j (\langle\psi\rangle + \frac{r}{c}\langle\frac{\partial\psi}{\partial t}\rangle) - r^{-1}\langle\frac{\partial\psi}{\partial n'}\rangle\}\, dS' \qquad (72)$$

upon using (64), where the pointed brackets denote evaluation at the retarded time as

$\langle f\rangle = f(x'_i, t - \frac{r}{c})$ for an arbitrary function f of position and time,

r is given by (57) and (58), and

n'_i is the outward unit normal with respect to the solid.

Equation (72) can be combined with (50) to produce

$$\phi(x_i, t) = 2\phi_{inc} + \frac{1}{2\pi}\int_S \{r^{-3} n'_j r_j (\langle\phi\rangle + \frac{r}{c}\langle\frac{\partial\phi}{\partial t}\rangle)\}\, dS' - \frac{1}{2\pi}\int_S \{r^{-1}\langle(u_{n'})_{solid}\rangle\}\, dS' \qquad (73)$$

for $\psi = \phi$ in the potential-based model, or with (51) to produce

$$p(x_i, t) = 2p_{inc} + \frac{1}{2\pi}\int_S \{r^{-3} n'_j r_j (\langle p\rangle + \frac{r}{c}\langle\frac{\partial p}{\partial t}\rangle)\}\, dS' + \frac{\rho}{2\pi}\int_S \{r^{-1}\langle(a_{n'})_{solid}\rangle\}\, dS' \qquad (74)$$

for $\psi = p$ in the pressure-based model. Methods which are based on the solution of either (73) or (74) are referred to as retarded potential methods. Computer code descriptions based on discretized versions of either (73) or (74) can be found in references [10-12].

APPROXIMATIONS TO RETARDED POTENTIAL METHOD

There are three common approximations to the retarded potential method which are referred to as the early-time approximation, the late-time approximation, and the DAA (Doubly Asymptotic Approximation).

Early-Time Approximation

If S is planar and p_{inc} is taken to be zero then (74) reduces to

$$p(x_i, t) = \frac{\rho}{2\pi}\int_S \{r^{-1}\langle (a_{n'})_{solid}\rangle\}\, dS' \tag{75}$$

since $n'_j r_j = 0$ everywhere, where n'_i is now a constant unit vector along S and S = (plane of z = constant). If, starting from zero, t represents the elapsed time and if the solid surface normal acceleration is zero for negative times then the response at any given point on S can be influenced only by points of S within a radius of ct. Equation (75) hence reduces further to

$$p(x, y, t) = \frac{\rho}{2\pi} n_j \int_0^{ct}\int_0^{2\pi} a_j(x + r\cos\theta, y + r\sin\theta, t - \frac{r}{c})\, d\theta dr \tag{76}$$

upon using $dS' = r dr d\theta$ with planar polar coordinates, where it is understood that a_i represents the solid surface acceleration. The early-time approximation assumes that

$$ct \ll L$$

where L is a characteristic length scale for S. The fact that $0 \le r \le ct$ bounds r leads to $r \ll L$, so that a Taylor series expansion gives

$$a_i(x + r\cos\theta, y + r\sin\theta, t - \frac{r}{c}) \approx a_i(x, y, t - \frac{r}{c}) + O(r)$$

for the a_i in (76). (Note that r/c is not small compared to t in this case and hence no expansion is warranted for the time dependence.) For small enough t, the radius ct becomes small enough that the initial assumption of a planar surface becomes true locally even for an S with global curvature, at least as long as the surface is smooth. Equation (76) now reduces to

$$p(x, y, t) = \rho n_j \int_0^{ct} a_j(x, y, t - \frac{r}{c})\, dr$$

since there is no longer any θ dependence in the integrand. The change of variable $t' = t - r/c$ transforms the above integral into

$$p(x, y, t) = \rho c n_j \int_0^t a_j(x, y, t')\, dt' , \tag{77}$$

which is one form of the early-time approximation. Differentiating (77) with respect to time gives another useful form

$$\frac{\partial p}{\partial t} = \rho c\, (a_n)_{solid} \tag{78}$$

on S for the early-time approximation. Higher order results for small ct can be found in reference [13].

Late-Time Approximation

The late-time approximation is based on the assumption

$$ct \gg L ,$$

which leads to $t \gg r/c$ because r is ultimately bound by $r \leq L$ in magnitude if L is the length scale associated with the largest global extent of S. (The above approximation can also be thought of in terms of the wavelength λ so that $c/\omega = \lambda \gg L$.) A Taylor expansion for small r/c for generic retarded functions f

$$\langle f \rangle = f(x'_i, t - \frac{r}{c}) \approx f(x'_i, t) - (\frac{r}{c})\frac{\partial f}{\partial t}(x'_i, t) + \frac{1}{2}(\frac{r}{c})^2\frac{\partial^2 f}{\partial t^2}(x'_i, t) + O(\frac{r}{c})^3 \tag{79}$$

is hence relevant in this case. Using (79) in (72) with zero for ψ_{inc} gives

$$\psi(x_i, t) = \frac{1}{2\pi}\int_S \{r^{-3}n'_j r_j \psi - r^{-1}\frac{\partial \psi}{\partial n'}\}\, dS' \tag{80}$$

upon retaining only lowest order terms. Equation (80) can be derived from Laplace's equation

$$\frac{\partial^2 \psi}{\partial x_j \partial x_j} = 0 \tag{81}$$

in direct analogy to the derivation of (72) from the wave equation (52) with no internal sources (so that ψ_{inc} and σ are zero). Equation (80) can hence be thought of as representing the governing equation on S for solutions to (81). If ψ is taken to be the velocity potential ϕ then (80) describes a potential flow and the flow is hence incompressible near the solid body [5, p. 280]. Equation (81), equation (50) on S, and the vanishing of the gradient of ϕ at infinity together form a boundary value problem for ϕ which can be reformulated in terms of a distributed surface source on S so as to ultimately produce an added mass model [14] of the fluid response. An alternative route to an added mass model is initiated by taking $\psi = p$ in (80) to get

$$p = \Gamma p - \rho^{-1}\Phi\frac{\partial p}{\partial n}, \tag{82}$$

where

$\Gamma(f) = \frac{1}{2\pi}\int_S \{r^{-3}n'_j r_j f\}\, dS'$ for Γ operating on a generic function f,

$\Phi(f) = \frac{\rho}{2\pi}\int_S (r^{-1}f)\, dS'$ for Φ operating on a generic function f, and

Γf is a shorthand notation for $\Gamma(f)$ and similarly for Φ.

The use of an operator notation, where the integrations of (80) are symbolically represented as the operators Γ and Φ in (82), is common in functional analysis and allows a great economy and conciseness in the development. Inserting (51) into (82) leads to

$$p = \Gamma p + \Phi(a_n)_{solid},$$

which can be symbolically solved for p as

$$p = \Lambda(a_n)_{solid} \tag{83}$$

where

I = the identity operator and

$\Lambda = (I-\Gamma)^{-1}\Phi$ for $(I-\Gamma)^{-1}$ as the inverse of the operator $(I-\Gamma)$.

The inverse T^{-1} of an operator T, when it exists, has the property

$$T^{-1}Tf = TT^{-1}f = f$$

for a generic function f, the Laplace transform and inverse transform being examples of such a T and T^{-1} respectively. An alternative, less abstract, interpretation consists of taking Γ and Φ as discretized versions of the corresponding integral operators of (80) so that Γ and Φ are then matrices and p, $\partial p/\partial n$, and the solid normal surface acceleration are then column vectors. Under such an interpretation, which would also be the practical approach in an actual implementation, equation (82) becomes an approximation to (80), the I becomes the identity matrix,

$(I-\Gamma)^{-1}$ becomes the inverse of the matrix $(I-\Gamma)$, and

Λ is the matrix multiplication of $(I-\Gamma)^{-1}$ and Φ.

The presentation in [14] is, in fact, given in terms of matrices originating from a finite element discretization. The force on the solid per unit area due to the fluid is $-pn_i$ and hence, by Newton's second law and (83), the Λ can be interpreted as an "added mass per unit area" operator (or matrix) which is due to the presence of the fluid. In other words, for an element at the surface S in a finite element subdivision of the solid, adding the fluid force acting on this element to the other such forces is equivalent to adding a mass operator to the mass of the solid for that element in the equations of motion.

Doubly Asymptotic Approximation

The Doubly Asymptotic Approximation (DAA) originated with Geers [15] and represents a smooth joining of the early-time and late-time (in the form of an added mass model) approximations. The simplest such joining consists of

$$\Lambda\frac{\partial p}{\partial t} + \rho c p = \rho c \Lambda (a_n)_{solid}, \tag{84}$$

so that (84) reduces to (78) when the $\partial p/\partial t$ term dominates and (84) reduces to (83) when the p term dominates. (It is assumed that the inverse of Λ exists.) Equation (84) is denoted by DAA_1, the next higher order version being denoted by DAA_2 [16]. An advantage of the DAA over the retarded potential method is that the DAA equation is local in time. This leads to substantial computational savings for the DAA when the methods are implemented numerically. The tradeoff is that the DAA is less accurate than the retarded potential method in a variety of circumstances [17].

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